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LIPSCHITZ AND ASYMPTOTIC STABILITY FOR NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate Lipschitz and asymptotic stability for nonlinear perturbed differential systems.

1. Introduction

The notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [9] . This notion of ULS lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation of Brauer[4] and uniformly stability in variation of Brauer and Strauss[3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. Furthermore, uniform Lipshitz stability neither implies asymptotic stability nor is it implied by it. Also, Elaydi and Farran[10] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Gonzalez and Pinto[11] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems.

The purpose of this paper is to employ the theory of integral inequalities to study Lipschitz and asymptotic stability for solutions of the nonlinear differential systems. The method incorporating integral inequalities takes an important place among the methods developed for

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the qualitative analysis of solutions to linear and nonlinear system of differential equations.

2. Preliminaries

We consider the nonlinear differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed differential system of (2.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds, \; y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we give some of the main definitions that we need in the sequel[9].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \le M|x_0|$ whenever $|x_0| \le \delta$ and $t \ge t_0 \ge 0$

(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \le M$ for $|x_0| \le \delta$ and $t \ge t_0 \ge 0$, (EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \delta$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq K\, e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

Remark 2.2. [11] The last definition implies that for $|x_0| \leq \delta$

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t.$$

We give some related properties that we need in the sequel.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.3. *Let x and y be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 2.4. [14] *Let $u, p, q, r, v \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, $w(u)$ be nondecreasing in u, and $u \leq w(u)$. Suppose that, for some $c \geq 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \Big(p(s) \int_{t_0}^{s} [q(\tau)u(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)w(u(a))da]d\tau\Big)ds,\ t \geq t_0. \tag{2.6}$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds\Big],\ t_0 \leq t < b_1, \tag{2.7}$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $u > 0$, $u_0 > 0$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.5. [12] *Let* $u, p, q, r, v \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c \geq 0$,

(2.8)

$$u(t) \leq c + \int_{t_0}^{t} \Big(p(s) \int_{t_0}^{s} \Big[q(\tau)w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a)w(u(a))da\Big]d\tau\Big)ds,\ t \geq t_0.$$

Then

(2.9)

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds\Big],\ t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.4 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.6. [8] *Let* $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u *and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$. *If , for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_1(s)\Big\{\int_{t_0}^{s} \lambda_2(\tau)w(u(\tau))d\tau\Big\}ds,\ t \geq t_0 \geq 0,$$

then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_2(s)ds\Big] \exp\Big(\int_{t_0}^{t} \lambda_1(s)ds\Big),\ t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.4 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda_2(s)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.7. [13] *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)(\int_{t_0}^{s} \lambda_3(\tau)u(\tau)d\tau)ds,\ 0 \leq t_0 \leq t.$$

Then

(2.10)

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds\Big],\ t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.4 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds \in \mathrm{domW^{-1}}\Big\}.$$

LEMMA 2.8. [6] *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\Big(\int_{t_0}^{s} \lambda_3(\tau) w(u(\tau))d\tau\Big)ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds\Big], \ t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.4 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau))ds \in \mathrm{domW^{-1}}\Big\}.$$

3. Main results

In this section, we investigate Lipschitz and asymptotic stability for solutions of the nonlinear perturbed differential systems.

THEOREM 3.1. *For the perturbed (2.2), we assume that*

$$\int_{t_0}^{t} |g(s, y(s))|ds \leq a(t)\Big(|y(t)| + \int_{t_0}^{t} k(s) w(|y(s)|)ds\Big),$$

where $a, k \in C(\mathbb{R}^+)$, $a, k, w \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty))$, $w(u)$ is nondecreasing in u, and $\frac{1}{v} w(u) \leq w(\frac{u}{v})$ for some $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + \int_{t_0}^{\infty} k(s)ds\Big] \exp\Big(\int_{t_0}^{\infty} M a(s)ds\Big), \tag{3.1}$$

where $M(t_0) < \infty$ and $b_1 = \infty$. Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.

Proof. Using the nonlinear variation of constants formula of Alekseev[1], the solutions of (2.1) and (2.2) with the same initial value are related by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\Big(\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big)ds.$$

Since $x = 0$ of (2.1) is ULSV, it is ULS by Theorem 3.3[9]. Using the ULSV condition of $x = 0$ of (2.1), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0|a(s)\frac{|y(s)|}{|y_0|}ds \\ &\quad + \int_{t_0}^{t} M|y_0|a(s)\int_{t_0}^{s} k(\tau)w\Big(\frac{|y(\tau)|}{|y_0|}\Big)d\tau ds.\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.6 yields

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + \int_{t_0}^{\infty} k(s)ds\Big] \exp\Big(\int_{t_0}^{\infty} Ma(s)ds\Big).$$

Thus we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. So, the proof is complete. □

THEOREM 3.2. *For the perturbed (2.2), we assume that*

$$|g(t,y)| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds,$$

where $a, b, k \in C(\mathbb{R}^+)$, $a, b, k, w \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty)$, $w(u)$ *is nondecreasing in* u, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau ds\Big], \tag{3.2}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Using the nonlinear variation of constants formula and the ULSV condition of $x = 0$ of (2.1), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))|d\tau ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} [a(\tau)w\Big(\frac{|y(\tau)|}{|y_0|}\Big)d\tau ds \\ &\quad + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} b(\tau)\int_{t_0}^{\tau} k(r)w\Big(\frac{|y(r)|}{|y_0|}\Big)dr]d\tau ds.\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.5 yields

$$|y(t)| \le |y_0| W^{-1}\Big[W(M) + M \int_{t_0}^{t}\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau ds\Big],$$

Thus we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

REMARK 3.1. Letting $b(t) = 0$ in Theorem 3.2, we obtain the same result as that of Theorem 3.1 in [5].

THEOREM 3.3. *For the perturbed (2.2), we assume that*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds,$$

where $a, b, k \in C(\mathbb{R}^+)$, $a, b, k, w \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty)$, $w(u)$ *is nondecreasing in* u, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big], \tag{3.3}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2.2) is ULS whenever the zero solution of (2.1) is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Applying Lemma 2.3, we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \Big| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \Big| ds \\ &\le M|y_0| + \int_{t_0}^{t} M|y_0|a(s)w(\frac{|y(s)|}{|y_0|})ds \\ &\quad + \int_{t_0}^{t} M|y_0|b(s)\int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{|y_0|})d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.8 yields

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big].$$

Hence we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

REMARK 3.2. Letting $b(t) = 0$ in Theorem 3.3, we obtain the same result as that of Theorem 3.3 in [5].

THEOREM 3.4. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbing term $g(t, y)$ satisfies*

$$|g(t, y(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds\Big), \tag{3.4}$$

where $\alpha > 0$, $a, b, k, w \in C(\mathbb{R}^+)$, $a, b, k, w \in L_1(\mathbb{R}^+)$, $w(u)$ is nondecreasing in u, $u \le w(u)$, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. If

(3.5)

$$M(t_0) = W^{-1}\Big[W(c) + \int_{t_0}^{\infty}\Big(Me^{\alpha s}\int_{t_0}^{s} a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau\Big)ds\Big] < \infty,\ t \ge t_0.$$

where $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, by remark 2.2, it is EVS. Using Lemma 2.3, we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big|ds \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}\Big[e^{-\alpha\tau}a(\tau)|y(\tau)| \\
&\quad +e^{-\alpha\tau}b(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|)drd\tau\Big]ds, \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}\Big[a(\tau)|y(\tau)|e^{\alpha\tau} \\
&\quad +b(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|e^{\alpha r})drd\tau\Big]ds.
\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. An application of Lemma 2.4 obtains

$$\begin{aligned}
|y(t)| &\le e^{-\alpha t}W^{-1}\Big[W(c) + \int_{t_0}^{t}(Me^{\alpha s}\int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau)ds\Big] \\
&\le e^{-\alpha t}M(t_0),\ t \ge t_0,
\end{aligned}$$

where $c = M|y_0|e^{\alpha t_0}$ and $M(t_0) > 0$. Therefore, all solutions of (2.2) approch zero as $t \to \infty$. $\square$

THEOREM 3.5. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbing term $g(t, y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le e^{-\alpha t}\Big(a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)|y(s)|ds\Big), \tag{3.6}$$

where $\alpha > 0$, $a, b, k, w \in C(\mathbb{R}^+)$, $a, b, k, w \in L_1(\mathbb{R}^+)$, $w(u)$ is nondecreasing in u and $u \le w(u)$. If

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] < \infty, b_1 = \infty, \tag{3.7}$$

where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, by remark 2.2, it is EVS. Using Lemma 2.3 and the assmptions, we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big| ds \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big[\frac{a(s)}{e^{\alpha s}}w(|y(s)|) \\
&\quad + \frac{b(s)}{e^{\alpha s}}\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau\Big] ds \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big[a(s)w(|y(s)|e^{\alpha s}) \\
&\quad + b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|e^{\alpha\tau}d\tau\Big] ds.
\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, an application of Lemma 2.7 obtains

$$\begin{aligned}
|y(t)| &\le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau\Big)ds\Big], \\
&\le e^{-\alpha t}M(t_0),\ t \ge t_0,
\end{aligned}$$

where $c = M|y_0|e^{\alpha t_0}$ and $M(t_0) > 0$. From the above estimation, we obtains the desired result. □

REMARK 3.3. Letting $b(t) = 0$ in Theorem 3.5, we obtain the same result as that of Theorem 3.7 in [5].

THEOREM 3.6. *Let the solution $x = 0$ of (2.1) be EASV. Suppose that the perturbing term $g(t, y)$ satisfies*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le e^{-\alpha t} a(t) \Big(|y(t)| + \int_{t_0}^{t} k(s) w(|y(s)|) ds \Big), \tag{3.8}$$

where $\alpha > 0$, $a, k, w \in C(\mathbb{R}^+)$, $a, k, w \in L_1(\mathbb{R}^+)$ and $w(u)$ is nondecreasing in u. If

$$M(t_0) = W^{-1}\Big[W(c) + \int_{t_0}^{\infty} k(\tau) d\tau\Big] \exp\Big(M \int_{t_0}^{\infty} a(s) ds\Big) < \infty, b_1 = \infty, \tag{3.9}$$

and $\int_{t_0}^{\infty} a(s) < \infty$ where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Using Lemma 2.3 and the assmptions, we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \Big| \int_{t_0}^{s} g(\tau, y(\tau)) d\tau \Big| ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M e^{-\alpha(t-s)} \Big[\frac{a(s)}{e^{\alpha s}} |y(s)| \\ &\quad + \frac{a(s)}{e^{\alpha s}} \int_{t_0}^{s} k(\tau) w(|y(\tau)|) d\tau \Big] ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M e^{-\alpha t} \Big[a(s)|y(s)|e^{\alpha s} \\ &\quad + a(s) \int_{t_0}^{s} k(\tau) w(|y(\tau)|e^{\alpha\tau}) d\tau \Big] ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, an application of Lemma 2.6 obtains

$$\begin{aligned} |y(t)| &\le e^{-\alpha t} W^{-1}\Big[W(c) + \int_{t_0}^{t} k(\tau) d\tau\Big] \exp\Big(M \int_{t_0}^{t} a(s) ds\Big), \\ &\le e^{-\alpha t} M(t_0),\ t \ge t_0, \end{aligned}$$

where $c = M|y_0|e^{\alpha t_0}$ and $M(t_0) > 0$. Therefore, all solutions of (2.2) approch zero as $t \to \infty$

□

REMARK 3.4. Letting $k(t) = 0$ in Theorem 3.6, we obtain the same result as that of Corollary 3.8 in [5].

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References

[1] V. M. Alekseev, *An estimate for the perturbations of the solutions of ordinary differential equations*, Vestn. Mosk. Univ. Ser. I. Math. Mekh. **2** (1961), 28-36(Russian).

[2] F. Brauer, *Perturbations of nonlinear systems of differential equations*, II, J. Math. Anal. Appl. **17** (1967), 418-434.

[3] F. Brauer and A. Strauss, *Perturbations of nonlinear systems of differential equations*, III, J. Math. Anal. Appl. **31** (1970), 37-48.

[4] F. Brauer, *Perturbations of nonlinear systems of differential equations*, IV, J. Math. Anal. Appl. **37** (1972), 214-222.

[5] S. I. Choi and Y. H. Goo, *Lipschitz and asymptotic stability for nonlinear differential systems*, preprint.

[6] S. K. Choi and N. J. Koo, *h-stability for nonlinear perturbed systems*, Ann. of Diff. Eqs. **11** (1995), 1-9.

[7] S. K. Choi, Y. H. Goo, and N. J. Koo, *Lipschitz and exponential asymptotic stability for nonlinear functional systems*, Dynamic Systems and Applications **6** (1997), 397-410.

[8] S. K. Choi, N. J. Koo, and S. M. Song, *Lipschitz stability for nonlinear functional differential systems*, Far East J. Math. Sci(FJMS)I **5** (1999), 689-708.

[9] F. M. Dannan and S. Elaydi, *Lipschitz stability of nonlinear systems of differential systems*, J. Math. Anal. Appl. **113** (1986), 562-577.

[10] S. Elaydi and H. R. Farran, *Exponentially asymptotically stable dynamical systems*, Appl. Anal. **25** (1987), 243-252.

[11] P. Gonzalez and M. Pinto, *Stability properties of the solutions of the nonlinear functional differential systems*, J. Math. Appl. **181** (1994), 562-573.

[12] Y. H. Goo, *Boundedness in the perturbed nonlinear differential systems*, Far East J. Math. Sci(FJMS)I **79** (2013), 205-217.

[13] Y. H. Goo, *Boundedness in the perturbed differential systems*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math **20** (2013), 223-232.

[14] Y. H. Goo, *Boundedness in perturbed nonlinear differential systems*, J. Chungcheong Math. Soc. **26** (2013), 605-613.

[15] Y. H. Goo, *Lipschitz and asymptotic stability for perturbed nonlinear differential systems*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **21** (2014), 11-21.

[16] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications Vol.I*, Academic Press, New York and London, 1969.

[17] B. G. Pachpatte, *A note on Gronwall-Bellman inequality*, J. Math. Anal. Appl. **44** (1973), 758-762.

[18] M. Pinto, *Integral inequalities of Bihari-type and applications*, Funkcial. Ekvac. **33** (1990), 387-404.

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